Rationality Theorems on D-finite Power Series

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A D-finite power series satisfies a system of linear partial differential equations with polynomial coefficients of special type. This class of power series has been systematically inverstigated by Stanley in the second volumn of his classic *Enumerative Combinatorics*. One of the central problems on power series is deciding whether a power series is rational, algebraic or D-finite? Two remarkable rationality theorems in this study are Szegő's theorem on power series with finite distinct coefficients and the Pólya–Carlson theorem on power series with integer coefficients. In this talk, we will present the multivariate extensions of these two theorems for D-finite power series.

Theorem 1. *Let K be a field of characteristic zero, and let Δ be a finite subset of K. Suppose that $f : \mathbb{N}^d \to \Delta$ with $d \geq 1$ is such that*

$$F(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in K[[x_1, \ldots, x_d]]$$

is D-finite. Then F is rational.

Theorem 2. *Let $F(x_1, \ldots, x_d) = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$ be a multivariate D-finite power series that converges on the open polydisc $B(0, 1)^d$. Then F is rational.*

As an application, we will show how these results can be used to study the nonnegative integer points on algebraic varieties.